Plane Tangent to a Surface

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Overview

We define the tangent plane at a point on a smooth surface in space.

We calculate an equation of the tangent plane from the partial derivatives of the function defining the surface.

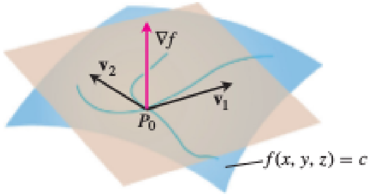
Tangent Planes and Normal Lines

If $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$ of this equation with respect to t leads to

$$\begin{aligned}
\frac{d}{dt} f(g(t), h(t), k(t)) &= \frac{d}{dt}(c) \\
\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + + \frac{\partial f}{\partial z}\frac{dk}{dt} &= 0 \\
\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right) . \left(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j} + \frac{dk}{dt}\boldsymbol{k}\right) &= 0 \\
\nabla f . \frac{d\boldsymbol{r}}{dt} &= 0.
\end{aligned}$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector

Tangent Planes and Normal Lines



Now let us restrict our attention to the curves that pass through P_0. All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curve's tangent lines all lie in the plane through P_0 normal to ∇f.

We call this plane the **tangent plane** of the surface at P_0. The line through P_0 perpendicular to the plane is the surface's normal line at P_0.

Definition

The tangent plane *at the point* $P_0(x_0, y_0, z_0)$ *on the level surface* $f(x, y, z) = c$ *of a differentiable function* f *is a plane through* P_0 *normal to* $\nabla f|_{P_0}$.
The normal line *of the surface at* P_0 *is the line through* P_0 *parallel to* $\nabla f|_{P_0}$.

Thus, the tangent plane and normal line have the following equations :

Tangent Plane to $f(x, y, z) = c$ at $P_0 = (x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

Normal Line to $f(x, y, z) = c$ at $P_0 = (x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t.$$

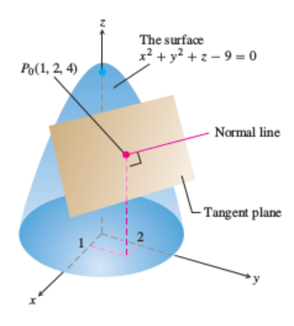
Tangent Plane and Normal Line : An Example

Example

The tangent plane and normal line to the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at $P_0(1, 2, 4)$ *are shown below.*



Plane Tangent to a Smooth Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$.

The surface $z = f(x, y)$ is therefore the zero level surface of the function

$$F(x, y, z) = f(x, y) - z.$$

The partial derivatives of F are $F_x = f_x,\ F_y = f_y,\ F_z = -1$.

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Tangent Line to the Curve of Intersection of Two Surfaces

Let

$$f(x, y, z) = c \quad \text{and} \quad g(x, y, z) = d$$

be two surfaces and let C be the curve of intersection of the surfaces.

The tangent line to C at $P_0(x_0, y_0, z_0)$ is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to

$$\boldsymbol{v} = \nabla f \times \nabla g = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}.$$

Hence the parametric equations of the tangent line to C at $P_0(x_0, y_0, z_0)$ is

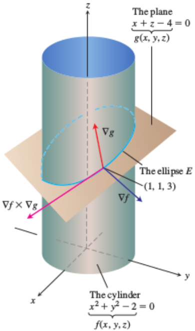
$$x = x_0 + v_1 t, \quad y = y_0 + v_2 t, \quad z = z_0 + v_3 t.$$

Tangent Line to the Curve of Intersection of Two Surfaces - An Example

The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ and $g(x, y, z) = x + z - 4 = 0$ meet in an ellipse E.

The line tangent to E at the point $P_0(1, 1, 3)$ is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to

$$\mathbf{v} = \nabla f \times \nabla g = 2\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}.$$



Hence the parametric equations for the tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

Exercises

1. *How do you find the tangent plane and normal line at a point on a level surface of a differentiable function $f(x, y, z)$? Give an example.*
2. *Find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0. Also, find parametric equations for the line that is normal to the surface at P_0.*
   - (a) $x^2 - y - 5z = 0,\ P_0(2, -1, 1)$
   - (b) $x^2 + y^2 + z = 4,\ P_0(1, 1, 2)$
3. *Find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point.*
   - (a) $z = \ln(x^2 + y^2),\ (0, 1, 0)$
   - (b) $z = 1/(x^2 + y^2),\ (1, 1, 1/2)$
4. *Find parametric equations for the line that is tangent to the curve of intersection of the surfaces at the given point.*
   - (a) *Surfaces:* $x^2 + 2y + 2z = 4,\ y = 1,$ *Point:* $(1, 1, 1/2)$
   - (b) *Surfaces:* $x + y^2 + z = 2,\ y = 1,$ *Point:* $(1/2, 1, 1/2)$

References

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